A Stroll Through Rindler Space

D.J. Martin, Ph.D.
D.J. Films-Multimedia, Inc.
Ottawa, Canada
(613) 523-4115

djfilms@djfilms.ca

Abstract: This paper presents a detailed analysis of Rindler space. Rindler space exists when a particle is accelerated such that its acceleration in its instantaneous rest frame is constant. There are many references which describe aspects of Rindler space but I have not found one which covers all of them. Nor have I found one with detailed, step-by-step derivations. So, here are the collected calculations, with no steps "left for the student to fill in".

This paper is still a work in progress. Only the first part of section 1 (Special Relativity) is complete, as of 16 November 2005.

Introduction

Section 1 performs calculations of Special Relativity (SR). Section 2 repeats the calculations using General Relativity (GR). The case of Rindler Space is interesting, because it is one of the few interesting cases involving acceleration that can be addressed by Special Relativity as well as General Relativity.

1. Special Relativity calculations

1.1 An Accelerated Observer

A particle moves from rest in an inertial frame G along the x-axis, so that it always has constant acceleration α relative to its instantaneous rest-frame. After time t in the inertial frame its instantaneous res-frame G' is moving at velocity v. For a particle moving with velocity u' in frame G',

$$\alpha = \frac{du'}{dt'} \tag{0.1}$$

In SR, collinear velocities add according to the following equation (with the speed of light c = 1):

$$u' = \frac{u - v}{1 - uv}$$

Differentiating this equation:

$$\mathrm{d}u' = \frac{\mathrm{d}u}{1 - uv} + (u - v)\mathrm{d}(1 - uv)$$

But the accelerated particle stays at the origin of G', hence u = v and so

$$\mathrm{d}u' = \frac{\mathrm{d}u}{1 - u^2} = \gamma^2 \mathrm{d}u \tag{0.2}$$

From the Lorentz transformation equations,

$$t = \gamma(t' + vx') .$$

Since the particle remains at the origin of the accelerating frame, $x' = 0$. Differentiating this equation then gives

$$\mathrm{d}t = \gamma \mathrm{d}t' \tag{0.3}$$

Inserting eqs. 1.2 and 1.3 into eq. 1.1,

$$\alpha = \frac{\gamma^2 \mathrm{du}}{\gamma^{-1}\mathrm{dt}} = \gamma^3 \frac{\mathrm{d}u}{\mathrm{d}t} \tag{0.4}$$

But,

$$\mathrm{d}[\gamma u] = \mathrm{d}[\left(1\text{-u}^2\right)^{-\frac{1}{2}} u] = -\frac{1}{2}\left(1\text{-u}^2\right)^{-\frac{3}{2}}.(-2u\mathrm{d}u).u + \left(1\text{-u}^2\right)^{-\frac{1}{2}} \mathrm{d}u$$

$$= u^2 \mathrm{d}u(1 - u^2)^{-\frac{3}{2}} + (1 - u^2)^{-\frac{1}{2}} \mathrm{d}u$$

$$= (1-u^2)^{-3/2}\mathrm{d}u.[u^2+1-u^2]$$
$$= \gamma^3\mathrm{d}u$$

Inserting this result into eq. 1.4, we have

$$\alpha\mathrm{d}t = \mathrm{d}[\gamma u]$$

Integrating with the particle starting from rest, u = 0 at time t = 0,

$$\alpha t = \gamma u = \frac{u}{(1-u^2)^{1/2}} \tag{0.5}$$

Rearranging this equation gives the speed of the particle, as observed by the inertial observer at time t in the inertial frame:

$$u = \frac{\alpha t}{(1+\alpha^2 t^2)} = \frac{\mathrm{d}x}{\mathrm{d}t} \tag{0.6}$$

But

$$d[(1+\alpha^2t^2)^{1/2}] = \frac{1}{2}(1+\alpha^2t^2)^{-\frac{1}{2}}\alpha^2 2t\,\mathrm{d}t$$
$$= \frac{\alpha^2 t\,\mathrm{d}t}{(1+\alpha^2t^2)^{1/2}}$$

Inserting this in eq. 1.6 gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}[(1+\alpha^2t^2)^{1/2}]$$

Whence

$$\mathrm{d}x = \frac{1}{\alpha}\mathrm{d}[(1+\alpha^2t^2)^{1/2}]$$

Integrating this from x = x_0 to x and t = 0 to t gives successively

$$x - x_0 = \frac{1}{\alpha}[(1+\alpha^2t^2)^{1/2} - 1]$$
$$\alpha(x-x_0)+1 = (1+\alpha^2t^2)^{1/2}$$
$$(\alpha(x-x_0)+1)^2 = (1+\alpha^2t^2)$$
$$-t^2 + x^2 - x(2x_0 - \frac{2}{\alpha}) = \frac{2x_0}{\alpha} - {x_0}^2$$

If the particle starts from $x_0 = 1/\alpha$, the equation simplifies to

$$-t^2 + x^2 = \frac{1}{\alpha^2} \tag{0.7}$$

This is the equation of a rectangular hyperbola in the (x, t) plane, as shown in figure 1 for $\alpha = 1$. The curve is the worldline of the accelerated particle.

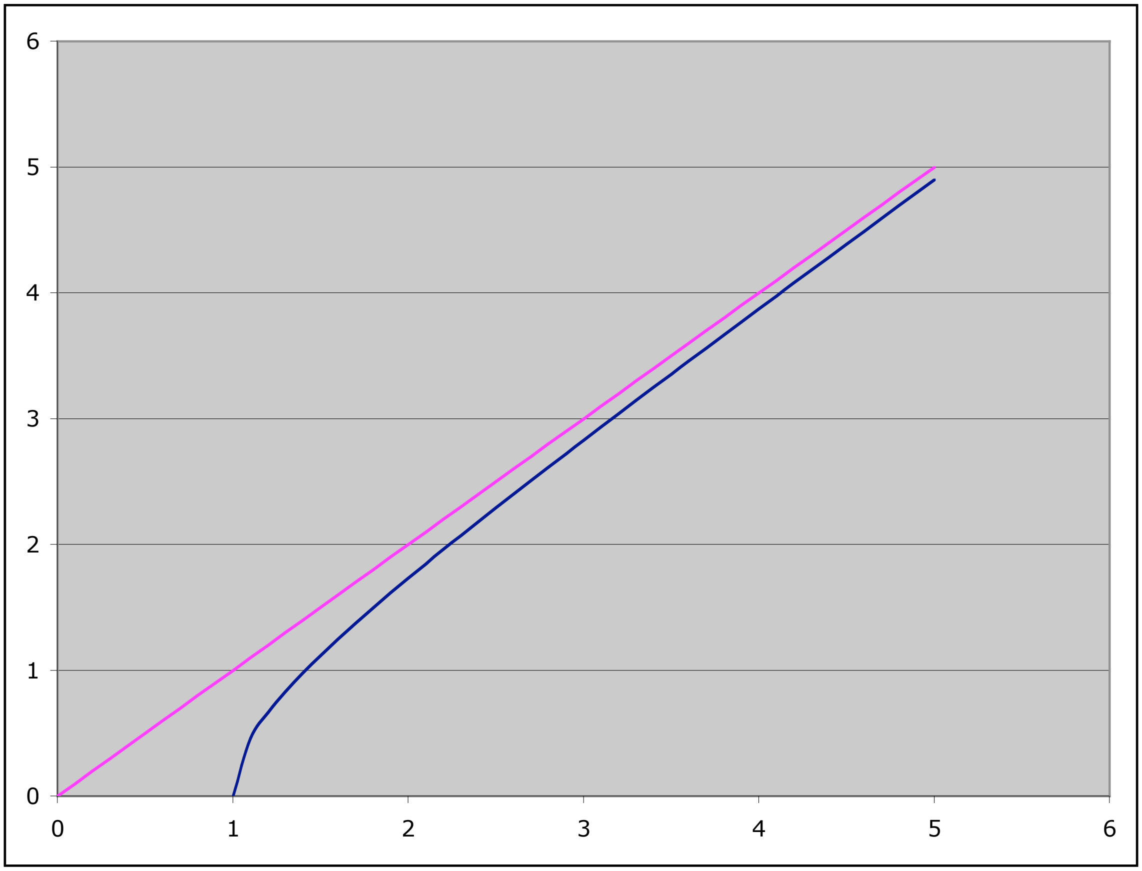

Figure 1.1. The worldline of an accelerated particle with a constant acceleration of 1, starting from x = 1 at time t = 0.

Note that the worldline is an asymptote to the line t = x, since

$$t = \sqrt{x^2 - 1}$$
$$\simeq x \quad \text{for} \quad x \gg 1$$

This leads to the first interesting fact about Rindler space: if a ray of light (whose worldline is at an angle of 45 degrees in the x-t plane) is emitted from the origin as the particle begins its acceleration, the ray of light will never catch up with the particle. Another aspect of this fact will surface when we examine the proper coordinates of the accelerated observer (the “horizon”).

1.2 Proper time and the accelerated particle

If a clock is carried with the accelerated particle, we can relate the proper time, τ, to the x, t and u values of the inertial observer, as follows.

From eq. 1.2 and the (constant) acceleration in the accelerated frame,

$$\alpha \mathrm{d}t' = \mathrm{d}u' = \gamma^2 \mathrm{d}u$$

$$\int_0^\tau \alpha \mathrm{d}t' = \int_0^u \frac{\mathrm{d}u}{1-u^2}$$

$$\alpha t = \tanh^{-1} u$$

$$u = \tanh(\alpha\tau) \tag{0.8}$$

Also,

$$\gamma(u) = (1-u^2)^{-\frac{1}{2}}$$

$$= (1-\tanh^2(\alpha\tau))^{-\frac{1}{2}}$$

$$= (\coth^2(\alpha\tau))^{-\frac{1}{2}}$$

$$\gamma(u) = \cosh(\alpha\tau) \tag{0.9}$$

Proper time is related to the inertial observer's time coordinate as follows, using eq. 1.5:

$$\alpha t = \alpha . \frac{\gamma u}{\alpha}$$

$$= \gamma u$$

$$= u \cosh(\alpha\tau)$$

$$= \tanh(\alpha\tau).\cosh(\alpha\tau)$$

$$\alpha t = \sinh(\alpha\tau) \tag{0.10}$$

Proper time is related to the inertial observer's x-coordinate as follows, using the relation derived above:

$$x - x_0 = \frac{1}{\alpha}[(1+\alpha^2 t^2)^{\frac{1}{2}} - 1]$$

$$= \frac{1}{\alpha}[(1+\sinh^2(\alpha\tau))^{1/2} - 1]$$

$$x - x_0 = \frac{1}{\alpha}[\cosh(\alpha\tau) - 1] \tag{0.11}$$

1.3 A Numerical Example

Rindler gives the following problem: Replace the particle by a space traveller T. if α is equal to the terrestrial acceleration g (to ensure maximum comfort), show that in 22 years by his own time T would cover $3.4x10^9$ light-years, about the distance of the farthest galaxies discernible with the 200-inch telescope. [Hint: using years and light-years as units, g = 1.03.] If, on the other hand, T describes a straight double path

ABCBA, with acceleration g on AB, CB and deceleration g on BC, BA for 6 years each, prove that in his 24 years' absence the earth will have aged by about 940 years. Show also that an unaccelerated year's cruise inserted in the middle of both the outward and return trips will add about another 480 years to the age of the earth on his return. (It should be noted that questions on this scale can be answered only approximately by special relativity; a rigorous solution must be based on a theory of cosmology.)

From eq. 1.11, with the traveler starting from the origin of the inertial coordinates:

$$x = \frac{1}{\alpha}[\cosh(\alpha\tau) - 1]$$
$$= \frac{1}{1.03}[\cosh(1.03\text{x}22) - 1]$$
$$x \simeq 3.3\text{x}10^9 \text{ light-years}$$

On each leg of the trip AB and CB,

$$\alpha t = \sinh(\alpha\tau)$$
$$1.03\, t = \sinh(1.03\text{x}6)$$
$$t_{AB} = 234.4 \text{ years}$$

By symmetry the total elapsed time, as seen by the inertial stay-at-home observer is

$$t_{elapsed} = 4\text{x}\,234.4$$
$$t_{elapsed} \simeq 937.8 \text{ years}$$

Adding an unaccelerated cruise of 1 year (ship-time) adds

$$\Delta t = \gamma\, \Delta\tau$$
$$= 241.5 \text{ years}$$

Therefore the two unaccelerated portions of the trip will add another 483 years to the journey.

1.4 The Space-time Diagram

Returning to figure 1.1, at an arbitrary time t, the tangent to the worldline is, by definition, the four-velocity of the particle. In the instantaneous rest-frame of the, this is the proper time axis, which is shown in figure 1.2. The gradient of the proper time axis on the space-time diagram is $\mathrm{d}t/\mathrm{d}x$. From the equation of the rectangular hyperbola, this gradient is ${x_0}/{t_0}$. The gradient of the proper x-axis (the line of simultaneity for the accelerated observer) is the reciprocal of this gradient. The equation of the line of simultaneity "y= mx + c" is, for a point P (x_0, t_0) is therefore

$$t = \frac{t_0}{x_0}x + c$$

Since the point $(x_{0,}\ t_0)$ also lies on the line of simultaneity, it satisfies this equation, and so the intercept c = 0. Therefore, for any point P on the worldline of the accelerated particle, the line of simultaneity (proper x-axis) for the instantaneous rest-frame is a straight line connecting the inertial observer's origin with the point P, as shown in figure 1.2.

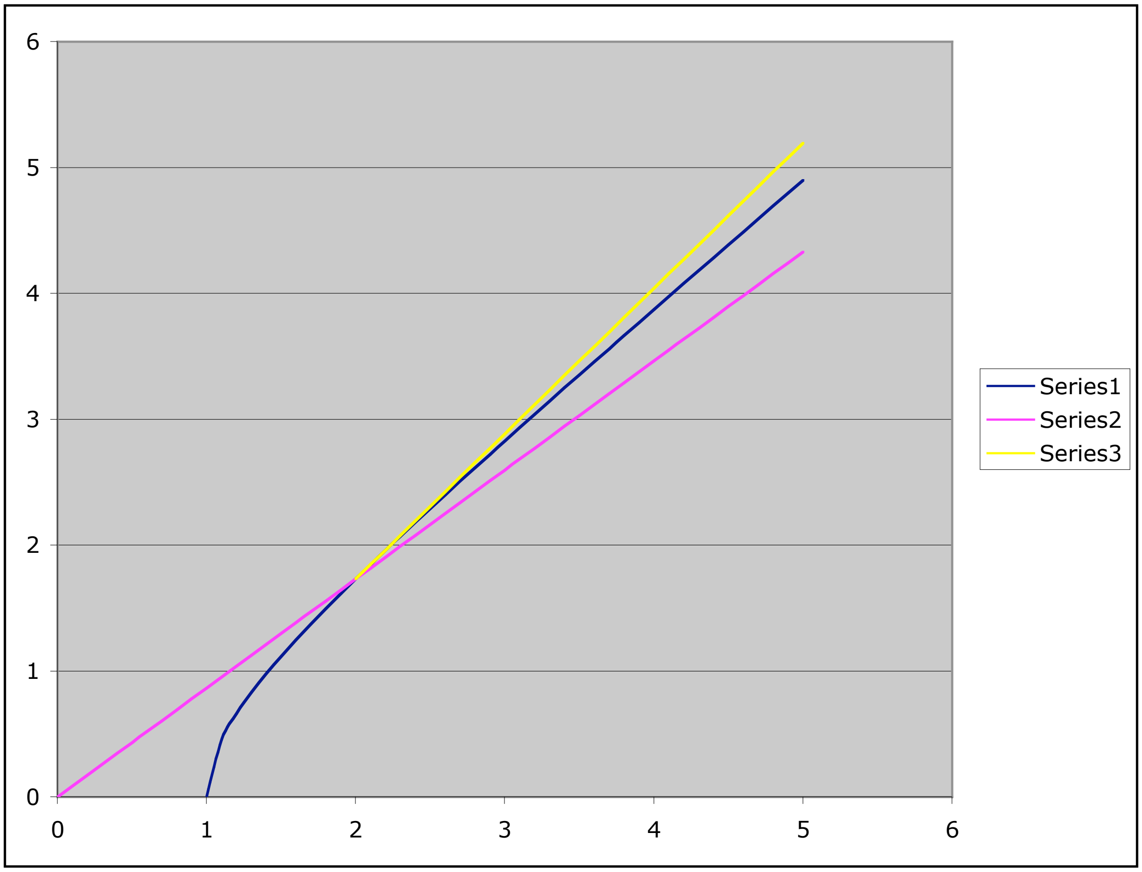

Fig. 1.2. Proper coordinate axes for the accelerated observer.

The next step is to calibrate these axes by finding the transformation equations from the coordinate of the inertial frame (x,t) to the coordinates (ξ, τ) of the instantaneous rest-frame of the accelerated particle. The derivation follows that of reference 1.

1.4 Coordinate Transformation between Inertial and Accelerated Frames

The proper coordinate system is described by functions $\tau(t,x)$ and $\xi(t,x)$ found by the following procedure. The accelerated observer's trajectory $t(\tau), x(\tau)$ corresponds to the line $\xi = 0$ in the proper coordinates.

Let the observer hold a rigid measuring stick of proper length ξ, so that the entire stick accelerates together with the observer. The stick is at rest in the co-moving frame and is described at an arbitrary time τ by the proper coordinates $(\tau, 0)$ [the end held by the observer] and (τ, ξ) [the far end]. If we find the coordinates (t, x) of the far end of the stick in the laboratory frame, we shall obtain the relation between the coordinates (t, x) and (τ, ξ).

4-VECTOR RE-DERIVATION

The comoving frame at proper time t is an inertial system of reference moving with the 4-velocity u' = dx'/dt, where xi' (t) is given by Eq. (8.3). [We need to consider a separate comoving frame for each proper time x.I The stick at time t can be repre- sented by the 4-vector s' (0,) connecting the events (t, 0) and (t,) in the comoving frame. The comoving frame is an inertial frame of reference, therefore the coordinates SI('lab) of the stick in the laboratory frame are found from the inverse Lorentz transfor- mation:

DIAGRAM TO GO HERE, TEXT TO BE EDITED ABOVE AND BELOW HERE.

Coordinates of Møller

The coordinate transformation which relates the inertial (t, x) coordinates to the Rindler (T,X) coordinates is given by Møller (ref 1) as:

$$\begin{aligned} at &= (1 + aX)\sinh aT \\ 1 + x &= (1 + aX)\cosh aT \end{aligned} \tag{0.12}$$

The worldline of the accelerated frame relative to the inertial frame can be plotted on an (t, x) diagram, by eliminating T in equations 1.1:

$$-a^2t^2 + (1 + ax)^2 = (1 + aX)^2(-\sinh^2 T + \cosh^2 T)$$

The last term in parentheses on the right side equals 1, and so the equation of the worldline of the accelerated frame is:

$$-a^2t^2 + (1+ax)^2 = (1+aX)^2 \tag{0.13}$$

For a coordinate X in the accelerated frame, the equation of the worldline is:

$$-t^2 + (x+\frac{1}{a})^2 = (X+\frac{1}{a})^2 \tag{0.14}$$

This curve is a rectangular hyperbola in the (t, x) plane, shown in figure 1 for X=0 (the origin in the accelerated frame), a=0.5, 1 and 2

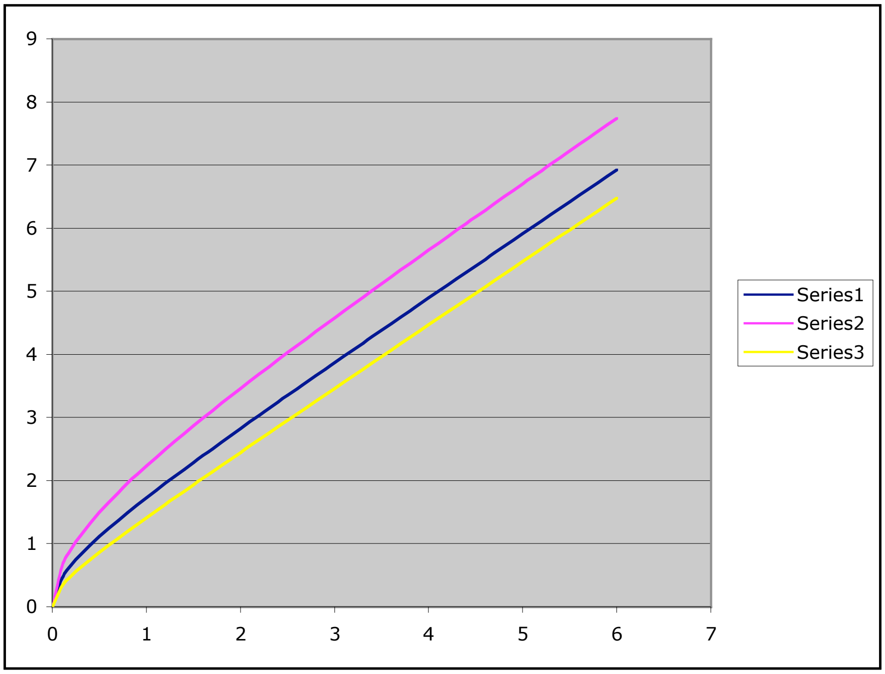

Figure 1: The worldline of the origin of an accelerated frame, as seen by an inertial observer, with a=0.5, 1 and 2.

The origin of the accelerated frame starts at t=x=T=0. The instantaneous velocity of the accelerated frame, relative to the inertial frame is found by differentiating eq. 1.3:

$$-2t\,dt + 2(x+\frac{1}{a})dx = 0$$

$$v = \frac{dx}{dt} = \frac{t}{(x+\frac{1}{a})} = \frac{\sqrt{(1+ax)^2-1}}{(1+ax)} \tag{0.15}$$

This is the inverse of the gradient of the tangent to the worldline. The tangent is also the worldline of a particle dropped from rest from the accelerated frame. The particle has an initial velocity given by eq 1.4, and travels on a straight line of the inverse of this gradient in an (x,t) plot. If the particle is dropped at time $t=\sqrt{3}$, and a=1 (with no loss of generality) then, from eq 1.3, x = 1. The equation of a straight line is "y = mx + c" where m = gradient and c is the intercept on the y axis. With the initial values of (x, t) = (1, $\sqrt{3}$), the gradient in the (x,t) plane is:

$$\frac{dt}{dx}=\frac{(x+1)}{t}=\frac{2}{\sqrt{3}}$$

and the equation of the worldline of the dropped particle is:

$$t=\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}} \tag{0.16}$$

This is plotted on figure 2.

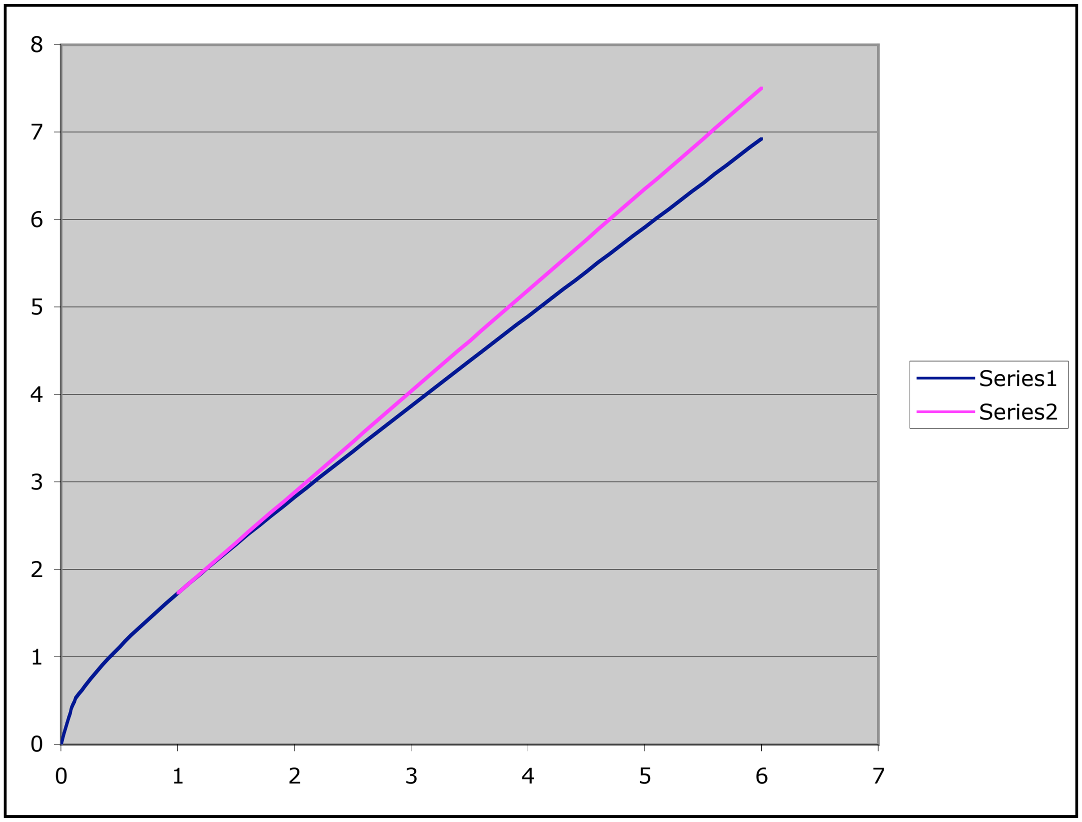

Figure 2: Plot of the worldlines of an accelerated observer and a particle dropped from rest from the accelerated frame at time $t=\sqrt{3}$

From the point of view of the inertial observer, I, the separation between the dropped particle and the origin of the accelerated frame A is:

$$\text{Separation at time } t=\left[sqrt(1+t^2)-1\right]-(t-\frac{1}{\sqrt{3}})\frac{\sqrt{3}}{2} \tag{0.17}$$

A plot of the separation seen by I is shown in figure 3.

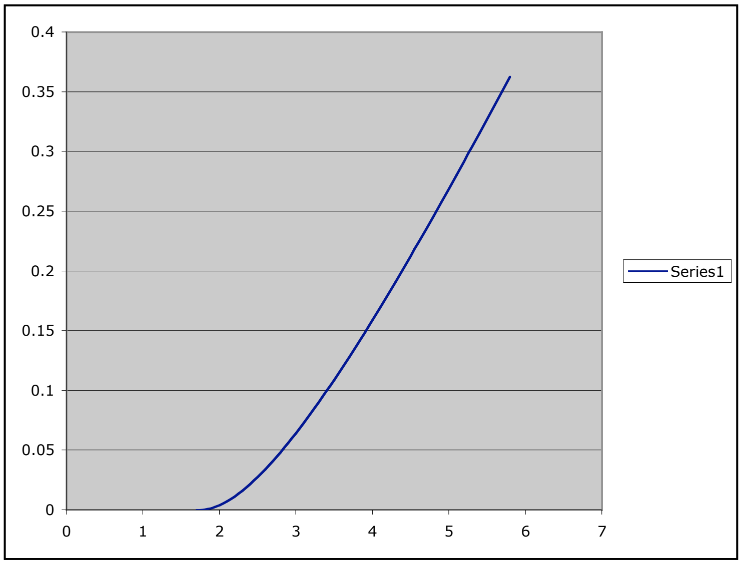

Figure 3: Plot of the separation between the dropped particle and A, as calculated by I

At large t, the separation is approximately proportional to t:

$$\text{Separation at time } t \simeq sqrt(t^2) - (t)\frac{\sqrt{3}}{2}$$

$$\simeq t(1 - \frac{\sqrt{3}}{2})$$

1.2 An Exploration of Lengths

From the coordinate transformation equations 1.1, when T=0,

$$t = (1 + aX)\sinh(0) = 0$$

$$1 + x = (1 + ax)\cosh(0) = 1 + aX = 1 \quad \text{for } X = 0$$

The subsequent analysis sets a=1 (without loss of generality).

At the start of the acceleration t=0 and T=0, and the origin (X=0) of A is at coordinate x=0.

At the start of the acceleration, the X=1 coordinate has x=1 Therefore, the unit length in A calculated by I is x=2 minus x=1, unity. This is not surprising, of course, as A is still at rest relative to I at t=T=0.

Figure 4 shows the space-time diagram after time t. The T axis is the line drawn from the origin of I through the point on the worldline of I at time t. The X axis is the line of simultaneity drawn through the same point. The T and X axes make equal angles with the line t=x (the worldline of a light ray). The equation for the X-axis is:

$$t=\frac{x}{\sqrt{3}}+\sqrt{3}-\frac{1}{\sqrt{3}} \tag{0.18}$$

The equation for the X=1 coordinate is:

$$t=\sqrt{(1+x)^{2}-4} \tag{0.19}$$

Equating these and solving for t, x gives:

$$\text{at Point } P_2: \quad x \simeq 2.09 \ \text{ and } t \simeq 2.36 \tag{0.20}$$

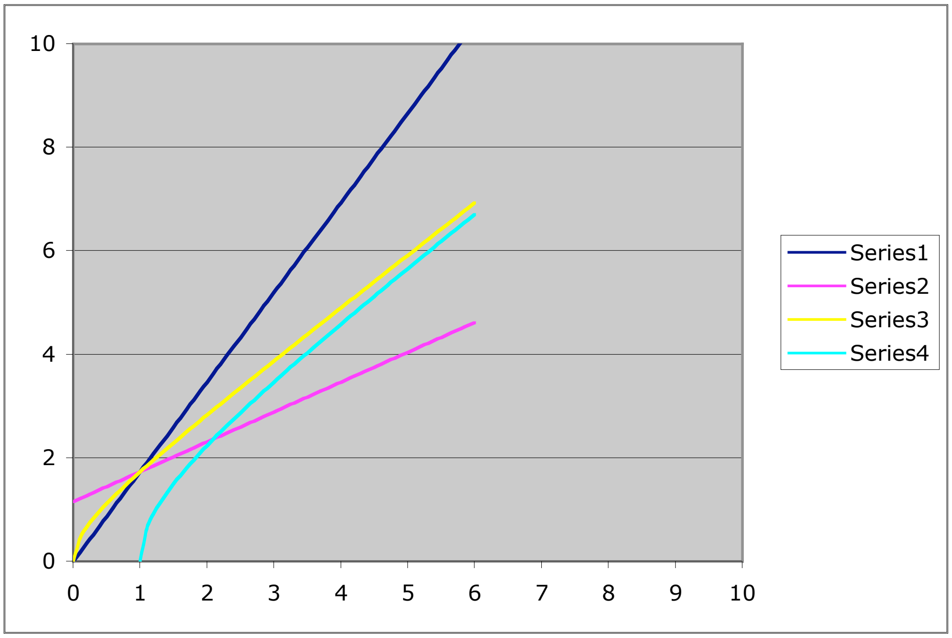

Fig. 4. Construction of the T- and X-axes for time $t=\sqrt{3}$

The point P_1 is the origin of the X, T coordinates. The point P_2 is the position of the X=1 mark after time t has elapsed. The distance between $X(P_2)$ and $X(P_1)$ measures the evolution of the unit length in (X, T) coordinates.

The point P_1 has coordinates ($\sqrt{3}$, 1) in (x,t) coordinates. The coordinates of point P2 in the inertial frame can be found from the intersection of the X-axis and the hyperbola with X=1:

$$\text{At point } \mathrm{P}_2: \quad t \simeq 2.36 \ \mathit{and} \ x \simeq 2.09$$

We can now calculate the distance between P1 and P2 in the I frame, using the I frame coordinates: TO BE VERIFIED?????????

$$\text{Distance P1-P2 :} \tag{0.21}$$

2. General Relativity calculations

The calculations of section 1will be repeated using General Relativity.